On Solution in Closed Form of Nonlinear Integral and Differential Equations of Fractional Order

Kilbas, Anatoly A.; Saigo, Megumi

数理解析研究所講究録
963巻 1996年 39-50

On Solution in Closed Form of Nonlinear Integral and Differential Equations of Fractional Order

Anatoly A. Kilbas[1] (ベラルーシ国立大学)

Megumi Saigo[2] [西郷 恵] (福岡大学理学部)

Abstract

Solutions in closed form of certain nonlinear integral equations and differential equations of fractional and integral order are given. Uniqueness of solutions of integral equations and applications to solving boundary value problems for differential equations are investigated.

1. Introduction

The paper is devoted to study the nonlinear Volterra integral equations

$$\varphi^m(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, dt + f(x) \quad (0 < x < d \leqq \infty) \tag{1.1}$$

for $\alpha > 0, m \in \mathbb{R}$ $(m \neq 0, 1)$, and the nonlinear differential equations of fractional order $\alpha > 0$

$$\left(D_{0+}^{\alpha} y\right)(x) = a(x) y^m(x) + f(x) \quad (0 < x < d \leqq \infty) \tag{1.2}$$

for $m \in \mathbb{R}$ $(m \neq 0, 1)$ with the Riemann-Liouville fractional derivative [23, Section 2]

$$\left(D_{0+}^{\alpha} y\right)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1} \frac{1}{\Gamma(1-\{\alpha\})} \int_0^x \frac{y(t)}{(x-t)^{\{\alpha\}}}\, dt \quad (\alpha > 0), \tag{1.3}$$

where $[\alpha]$ and $\{\alpha\}$ are integral and fractional parts of α, respectively.

---

[1]Department of Mathematics and Mechanics, Belarusian State University, Minsk 220050, Belarus
[2]Department of Applied Mathematics, Fukuoka University, Fukuoka 814-80, Japan

The equation (1.1) being arisen in the nonlinear theory of wave propagation [13] and water perlocation [8], [21] belongs to Abel's type integral equations [9], [23] and contains the Riemann-Liouville fractional integral [23, Section 2]

$$(1.4) \qquad \left(I_{0+}^{\alpha}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\,dt \quad (\alpha > 0).$$

Therefore we call (1.1) the integral equation of fractional order.

The equation (1.1) with $m > 0$ and the equation

$$(1.5) \qquad \varphi^m(x) = a(x)\int_0^x k(x-t)\varphi(t)dt + f(x) \quad (0 < x < d \leqq \infty)$$

for $\alpha > 0, m \in \mathbb{R}$ $(m \neq 0, 1)$ with the convolution kernel $k(x-t)$ were studied in [1], [4], [6], [10], [19], [20], [21] for $a(x) = 1$ and in [2], [3], [5], [7] in general case. These papers in the main were devoted to study the existence and uniqueness for the solution $\varphi(x)$ of the nonhomogeneous equation (1.5) with $m > 1$, the stability of such a solution and the method of successive approximation to constract this solution. Some results of such a type for the nonlinear equation (1.5) with $a(x) = 1$ were obtained in [1], [4] and [10] for $0 < m < 1$, in [12] for $m < -1$, and for the equation (1.1) with $m > 0$ in [14]. We also note that in [15], [16], [17] and [22] we investigated asymptotic behavior of the solution $\varphi(x)$ of the equation (1.1) at zero, provided that $a(x)$ and $f(x)$ have special power asymptotics at zero and in [11] we found the first term of the asymptotics of the solution $\varphi(x)$ of the equation (1.5) at zero in the case when $a(x), k(x)$ and $f(x)$ have power asymptotics at zero. Problems of existence and uniqueness of the solutions of Cauchy type and Dirichlet type for nonlinear differential equations of fractional order are also studied by many authors (see [23, Sections 42-43] and [18]). Explicit solutions are known only for the simplest, basically linear, fractional integral and differential equations (1.1) and (1.2).

The paper deals with solution in closed form of the nonlinear fractional integral and differential equations (1.1) and (1.2) with $a(x) = ax^l$ $(a, l \in \mathbb{R},\ a \neq 0)$ and monomial free term $f(x) = bx^n$ $(b, n \in \mathbb{R})$. Section 2 is devoted to obtain the explicit solutions of nonhomogeneous and homogeneous $(f(x) = 0)$ integral equations. In Section 3 we give solutions in closed form of nonhomogeneous and homogeneous differential equations of fractional order, in Section 4 of the corresponding ordinary differential equations. Section 5 deals with studying the uniqueness of the obtained solutions of integral equations. In Section 6 we discuss applications to solve the boundary value problems for differential equations.

2. Solution of Nonlinear Integral Equations

We consider the nonlinear integral equation (1.1) with $a(x) = ax^l$ and $f(x) = bx^n$ for $a, b, l, n \in \mathbb{R}$ $(a \neq 0, b \neq 0)$:

$$(2.1) \qquad \varphi^m(x) = \frac{ax^l}{\Gamma(\alpha)}\int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\,dt + bx^n \quad (0 < x < d \leqq \infty).$$

with $m \in \mathbb{R}$ $(m \neq 0, 1)$ and $\alpha > 0$. We shall seek a solution $\varphi(x)$ of the equation (2.1) in the form

$$\varphi(x) = cx^{\beta}. \tag{2.2}$$

Then according to (1.4) and the relation in [23, (2.44)]

$$\left(I_{0+}^{\alpha} t^{\beta}\right)(x) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta} \tag{2.3}$$

for $\beta > -1$. We suppose that $\beta m = l + \alpha + \beta = n$ and that the equation

$$\xi^{m} - \frac{a\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\xi - b = 0 \tag{2.4}$$

is solvable with $\xi = c$ being its solution. Then it is directly verified that (2.2) gives the solution of the equation (2.1). From here we arrive at the following statements.

Theorem 1. *Let $\alpha > 0$, $a, b, m \in \mathbb{R}$ $(a, b \neq 0; m \neq 0, 1)$ and $\beta > -1$. Let the equation (2.4) with $a, b \in \mathbb{R}$ $(a, b \neq 0)$ be solvable with $\xi = c$ being its solution. Then the nonlinear integral equation*

$$\varphi^{m}(x) = \frac{ax^{-\alpha+(m-1)\beta}}{\Gamma(\alpha)} \int_{0}^{x} \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, dt + bx^{m\beta} \quad (0 < x < d \leqq \infty) \tag{2.5}$$

is solvable and its solution $\varphi(x)$ has the form (2.2).

Corollary 1.1. *Let $\alpha > 0$ and $a, b, m \in \mathbb{R}$ $(a, b \neq 0; m \neq 0, 1)$. Let the equation*

$$\xi^{m} - \frac{a}{\Gamma(\alpha+1)}\xi - b = 0 \tag{2.6}$$

is solvable and let $\xi = c$ be its solution. Then the nonlinear integral equation

$$\varphi^{m}(x) = \frac{ax^{-\alpha}}{\Gamma(\alpha)} \int_{0}^{x} \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, dt + b \quad (0 < x < d \leqq \infty) \tag{2.7}$$

is solvable and its solution $\varphi(x)$ is the constant:

$$\varphi(x) = c. \tag{2.8}$$

Remark 1. The solvability of the equation (2.5) depends on that of the algebraic equation (2.6). As it was proved in [11], the latter equation can have one or two positive solutions for $m > 0$.

Now we consider the homogeneous nonlinear integral equation

$$\varphi^m(x) = \frac{ax^{-\alpha+(m-1)\beta}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, dt \quad (0 < x < d \leqq \infty) \tag{2.9}$$

corresponding to the equation (2.5) provided that the conditions in Theorem 1 are valid. The direct calculation proves that the function

$$\varphi(x) = \left[\frac{\Gamma(\beta+1)a}{\Gamma(\alpha+\beta+1)}\right]^{1/(m-1)} x^\beta \tag{2.10}$$

gives an exact solution of the equation (2.9). Then, setting $l = -\alpha + (m-1)\beta$, we come to the result:

Theorem 2. *Let $\alpha > 0$, $a, m, l \in \mathbb{R}$ $(m \neq 0, 1)$ such that*

$$\frac{l+\alpha}{m-1} > -1. \tag{2.11}$$

Then the homogeneous nonlinear integral equation

$$\varphi^m(x) = \frac{ax^l}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, dt \quad (0 < x < d \leqq \infty) \tag{2.12}$$

is solvable and its solution $\varphi(x)$ has the form

$$\varphi(x) = \left\{\frac{\Gamma(\{(l+\alpha)/(m-1)\}+1)a}{\Gamma(\alpha+\{(l+\alpha)/(m-1)\}+1)}\right\}^{1/(m-1)} x^{(l+\alpha)/(m-1)}. \tag{2.13}$$

Corollary 2.1. *If $\alpha > 0$, $a, m \in \mathbb{R}$ $(m \neq 0, 1)$, then the homogeneous nonlinear integral equation*

$$\varphi^m(x) = \frac{ax^{-\alpha}}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, dt \quad (0 < x < d \leqq \infty), \tag{2.14}$$

is solvable and its solution $\varphi(x)$ is given by

$$\varphi(x) = \left\{\frac{a}{\Gamma(\alpha+1)}\right\}^{1/(m-1)}. \tag{2.15}$$

Remark 2. In particular, the equation (2.12) with $m > 1$ and $a = \Gamma(\alpha+\beta+1)/\Gamma(\beta+1)$ arose in the heat theory and its solution was obtained in [24]. For the case $l = 0$ and $a = \Gamma(\alpha)$ with $m < -1$ the solution of the equation (2.12) was given in [12].

3. Solution of Nonlinear Differential Equations of Fractional Order

Now we consider the nonlinear differential equation (1.2) with $a(x) = ax^l$ and $f(x) = bx^n$ with $a, b, l, n \in \mathbb{R}$ $(a \neq 0; b \neq 0)$:

$$(D_{0+}^{\alpha} y)(x) = ax^l y^m(x) + bx^n \quad (0 < x < d \leqq \infty) \tag{3.1}$$

for $m \in \mathbb{R}$ $(m \neq 0, 1), \alpha > 0$. Seeking the solution $\varphi(x)$ of the equation (3.1) in the form

$$y(x) = Cx^{\gamma} \tag{3.2}$$

by using the relation

$$\left(D_{0+}^{\alpha} t^{\gamma}\right)(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-\alpha} \tag{3.3}$$

for $\gamma > -1$ (see [23, (2.26)]), similar arguments to Theorem 1 imply that

Theorem 3. *Let $\alpha > 0$, $a, b, m \in \mathbb{R}$ $(a, b \neq 0; m \neq 0, 1)$ and $\gamma > -1$. Let the equation*

$$a\xi^m - \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}\xi + b = 0 \tag{3.4}$$

be solvable with $\xi = C$ being its solution. Then the nonlinear differential equation of fractional order

$$\left(D_{0+}^{\alpha} y\right)(x) = ax^{(1-m)\gamma-\alpha} y^m(x) + bx^{\gamma-\alpha} \quad (0 < x < d \leqq \infty) \tag{3.5}$$

is solvable and its solution $y(x)$ has the form (3.2).

Corollary 3.1. *Let $\alpha > 0$, $a, b, m \in \mathbb{R}$ $(a, b \neq 0; m \neq 0, 1)$ and $k = 1, 2, \cdots, -[-\alpha]$. Then the nonlinear differential equation of fractional order*

$$\left(D_{0+}^{\alpha} y\right)(x) = ax^{m(k-\alpha)-k} y^m(x) + bx^{-k} \quad (0 < x < d \leqq \infty) \tag{3.6}$$

is solvable and its solution $y(x)$ is given by

$$y(x) = \left(-\frac{b}{a}\right)^{1/m} x^{\alpha-k}. \tag{3.7}$$

Corollary 3.1 follows from Theorem 3 by setting $\gamma = \alpha - k$ $(k = 1, 2, \cdots, -[-\alpha])$ and taking into account the relation (see [23, (1.57)])

$$\lim_{z \to -k} \frac{1}{\Gamma(z)} = 0 \quad (k = 0, 1, 2, \cdots). \tag{3.8}$$

Corollary 3.2. *Let $\alpha > 0$ $(\alpha \neq 1, 2, \cdots)$ and $a, b, m \in \mathbb{R}$ $(a, b \neq 0; m \neq 0, 1)$. Let the equation*

$$a\xi^m - \frac{\xi}{\Gamma(1-\alpha)} + b = 0 \tag{3.9}$$

be solvable with $\xi = C$ being its solution. Then the nonlinear differential equation of fractional order

$$\left(D_{0+}^{\alpha} y\right)(x) = a x^{-\alpha} y^m(x) + b x^{-\alpha} \quad (0 < x < d \leqq \infty) \tag{3.10}$$

is solvable and its solution is the constant:

$$y(x) = C. \tag{3.11}$$

Theorem 4. *Let $\alpha > 0$, $a, m, l \in \mathbb{R}$ $(a \neq 0; m \neq 0, 1)$ such that*

$$\frac{l+\alpha}{1-m} > -1 \tag{3.12}$$

and

$$\frac{l+\alpha}{1-m} \neq \alpha - k \quad (k = 1, 2, \cdots, -[-\alpha]). \tag{3.13}$$

Then the homogeneous nonlinear differential equation of fractional order

$$\left(D_{0+}^{\alpha} y\right)(x) = a x^l y^m(x) \quad (0 < x < d \leqq \infty) \tag{3.14}$$

is solvable and has the nonzero solution $y(x)$ of the form

$$y(x) = \left\{ \frac{\Gamma(\{(l+\alpha)/(1-m)\} + 1)}{\Gamma(\{(l+\alpha)/(1-m)\} - \alpha + 1)a} \right\}^{1/(m-1)} x^{(l+\alpha)/(1-m)}. \tag{3.15}$$

Remark 3. If the condition (3.13) does not hold, namely if there exists $k = 1, 2, \cdots, -[-\alpha]$ such that

$$\frac{l+\alpha}{1-m} = \alpha - k, \tag{3.16}$$

then in view of (3.8) the equation (3.14) has only the trivial solution.

Corollary 4.1. *If $\alpha > 0$ $(\alpha \neq 1, 2, \cdots)$ and $a, m \in \mathbb{R}$ $(a \neq 0; m \neq 0, 1)$, then the homogeneous nonlinear differential equation of fractional order*

$$\left(D_{0+}^{\alpha} y\right)(x) = a x^{-\alpha} y^m(x) \quad (0 < x < d \leqq \infty) \tag{3.17}$$

is solvable and has the nonzero constant solution:

$$y(x) = \{a\Gamma(1-\alpha)\}^{1/(1-m)}. \tag{3.18}$$

4. Solution of Nonlinear Ordinary Differential Equations

When $\alpha = n = 1, 2, \cdots$, the equations (3.5) and (3.14) become the ordinary differential equations

$$y^{(n)}(x) = ax^{(1-m)\gamma-n}y^m(x) + bx^{\gamma-n} \quad (0 < x < d \leqq \infty), \tag{4.1}$$

$$y^{(n)}(x) = ax^l y^m(x) \quad (0 < x < d \leqq \infty), \tag{4.2}$$

and from Theorems 3 and 4 we arrive at the following results.

Theorem 5. *Let $n = 1, 2, \cdots$, $a, b, m \in \mathbb{R}$ $(a, b \neq 0; m \neq 0, 1)$ and $\gamma > -1$. Let the equation*

$$a\xi^m - \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-n+1)}\xi + b = 0 \tag{4.3}$$

be solvable with $x = C$ being its solution. Then the nonlinear differential equation (4.1) is solvable and its solution $y(x)$ has the form (3.2).

Corollary 5.1. *Let $n = 1, 2, \cdots$; $k = 0, 1, \cdots, n-1$ and $a, b, m \in \mathbb{R}$ $(a, b \neq 0; m \neq 0, 1)$. Then the nonlinear differential equation*

$$y^{(n)}(x) = ax^{(1-m)k-n}y^m(x) + bx^{k-n} \quad (0 < x < d \leqq \infty) \tag{4.4}$$

is solvable and its solution $y(x)$ is given by

$$y(x) = \left(-\frac{b}{a}\right)^{1/m} x^k. \tag{4.5}$$

In particular, the solution of the equation

$$y^{(n)}(x) = ax^{-n}y^m(x) + bx^{-n} \quad (0 < x < d \leqq \infty) \tag{4.6}$$

is the constant

$$y(x) = \left(-\frac{b}{a}\right)^{1/m}. \tag{4.7}$$

Theorem 6. *Let $n = 1, 2, \cdots$ and $a, m, l \in \mathbb{R}$ $(a \neq 0; m \neq 0, 1)$ be such that*

$$\frac{l+n}{1-m} > -1, \quad \frac{l+n}{1-m} \neq k \quad (k = 0, 1, \cdots, n-1). \tag{4.8}$$

Then the homogeneous nonlinear differential equation

$$y^{(n)}(x) = ax^l y^m(x) \quad (0 < x < d \leqq \infty) \tag{4.9}$$

is solvable and has the nonzero solution of the form

$$y(x) = \left\{ \frac{\Gamma(\{(l+n)/(1-m)\}+1)}{a\Gamma(\{(l+n)/(1-m)\}-n+1)} \right\}^{1/(m-1)} x^{(l+n)/(1-m)}. \tag{4.10}$$

5. Uniqueness of Solutions of Nonlinear Integral Equations

To investigate the uniqueness of the solutions of the nonlinear integral equations (2.5) and (2.12), given in Section 2, we use the results from [14]. For $0 < d < \infty$ we denote by $C(0, d)$ the space of functions continuous on $(0, d)$. Let $C^+(0, d)$ be subspace of $C(0, d)$ consisting of nonnegative functions, and let $C^+_\varepsilon(0, d)$ be the subspace of $C^+(0, d)$ consisting of functions $g(x) \geqq 0$ for which there exists a constant $\varepsilon = \varepsilon(g) > 0$ such that $g(x) \geqq \varepsilon$ for $x \in (0, d)$. The following asertions about the uniqueness of the solutions $\varphi(x)$ of the equation (1.1) and the corresponding homogeneous equation

$$\varphi^m(x) = \frac{a(x)}{\Gamma(\alpha)} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, dt \quad (0 < x < d \leqq \infty) \tag{5.1}$$

for $\alpha > 0, m \in \mathbb{R}$ $(m \neq 0, 1)$ follow from the results in [14].

Lemma 1. *Let $\alpha > 0$, $0 < m \leqq 1$ and $0 < d \leqq \infty$.*

(i) *If $a(x)$, $f(x) \in C(0, d)$ [or $C^+(0, d)$] and the equation (1.1) has a solution in the space $C(0, d)$ [or $C^+(0, d)$], then the solution is unique.*

(ii) *If $a(x) \in C(0, d)$ [or $C^+(0, d)$] and the equation (5.1) has a solution in the space $C(0, d)$ [or $C^+(0, d)$], then the solution is unique.*

Lemma 2. *Let $\alpha > 0$, $m > 1$ and $0 < d \leqq \infty$.*

(i) *If $a(x) \in C^+(0, d)$, $f(x) \in C^+_\varepsilon(0, d)$ and the equation (1.1) has a solution in the space $C^+_0(0, d)$, then the solution is unique.*

(ii) *If $a(x) \in C^+(0, d)$ and the equation (5.1) has a solution in the space $C^+_\varepsilon(0, d)$, then the solution is unique.*

Using solutions (2.2) and (2.13) and applying Lemmas 1-2, we obtain the following results.

Theorem 7. *Let $\alpha > 0$ and $a, b, m, \beta \in \mathbb{R}$ $(a, b \neq 0; m \neq 0, 1)$ and let $\xi = c$ be the unique solution of the equation* (2.4).

(i) *If $0 < m < 1$ and $\beta > -1$, then* (2.2) *is the unique solution of the equation* (2.5) *in the space $C(0, d)$. If additionally $a > 0$, $b > 0$ and $c > 0$, then this solution belongs to the space $C^{+}(0, d)$.*

(ii) *If $m > 1$, $-1 < \beta < 0$, $a > 0$, $b > 0$, $c > 0$ and $0 < d < \infty$, then* (2.2) *is the unique solution of the equation* (2.5) *in the space $C_{\varepsilon}^{+}(0, d)$* with $\varepsilon = d^{\beta}$.

Theorem 8. *Let $\alpha > 0$ and $a, m, l \in \mathbb{R}$ $(a \neq 0; m \neq 0, 1)$.*

(i) *If $0 < m < 1$ and $l + \alpha < 1 - m$, then* (2.13) *is the unique solution of the equation* (2.12) *in the space $C(0, d)$. If additionally $a > 0$, then this solution belong to the space $C^{+}(0, d)$.*

(ii) *If $m > 1$, $1 - m < l + \alpha < 0$, $a > 0$, $b > 0$ and $0 < d < \infty$, then* (2.13) *is the unique solution of the equation* (2.12) *in the space $C_{\varepsilon}^{+}(0, d), \varepsilon = d^{\beta}$.*

6. Applications to Boundary Value Problems for Differential Equations

The results, given in Sections 3 and 4, can be applied to solve the boundary value problems for the nonlinear differential equations of fractional and integeral order. For example, the following results follow from Corollaries 3.1 and 5.1.

Theorem 9. *Let $\alpha > 0$, $a, b, m \in \mathbb{R}$ $(m \neq 0, 1)$, $n = -[-\alpha]$ and let k be an integer such that $1 \leqq k \leqq n$. Then the Cauchy type boundary value problem for the nonlinear differential equation of fractional order*

$$\left(D_{0+}^{\alpha} y\right)(x) = a x^{m(k-\alpha)-k} y^{m}(x) + b x^{-k} \quad (0 < x < d \leqq \infty); \tag{6.1}$$

with

$$\begin{aligned} &\left(D_{0+}^{\alpha-j} y\right)(0) = 0 \quad (j - 1, 2, \cdots, n;\ j \neq k), \\ &\left(D_{0+}^{\alpha-k} y\right)(0) = \Gamma(\alpha - k + 1)\left(-\frac{b}{a}\right)^{1/m} \end{aligned} \tag{6.2}$$

is solvable and its solution $y(x)$ has the form

$$y(x) = \left(-\frac{b}{a}\right)^{1/m} x^{\alpha-k}. \tag{6.3}$$

Corollary 9.1. *Let $n = 1, 2, \cdots$, $a, b, m \in \mathbb{R}$ $(m \neq 0, 1)$. Then the Cauchy problem for the nonlinear differential equation*

$$y^{(n)}(x) = a x^{(1-m)k-n} y^{m}(x) + b x^{k-n} \quad (0 < x < d \leqq \infty); \tag{6.4}$$

with

$$y^{(j)}(0) = 0 \quad (j \neq k), \quad y^{(k)}(0) = k! \left(-\frac{b}{a}\right)^{1/m} \tag{6.5}$$

is solvable and its solution $y(x)$ has the form

$$y(x) = \left(-\frac{b}{a}\right)^{1/m} x^k. \tag{6.6}$$

The uniqueness problem of the solutions (6.3) and (6.6) for the boundary value problems (6.1) - (6.2) and (6.4) - (6.5) is more comlicated than for the integral equations. For example, we can not prove even the uniqueness of the solution $y(x)$ given in (6.3) from the known results for nonlinear differential equations of fractional order.

Indeed, it is known [23, section 42.1] that the Cauchy type problem for the nonlinear differential equation of fractional order $\alpha > 0$

$$\left(D_{0+}^{\alpha} y\right)(x) = f(x, y); \quad (n - 1 < \alpha \leqq n, \ n = -[-\alpha]); \tag{6.7}$$

with initial conditions

$$\left(D_{0+}^{\alpha-k} y\right)(0) = b_k \quad (k = 1, 2, \cdots, n) \tag{6.8}$$

has a unique continuous solution $y(x)$ in the open interval $D \subset \mathbb{R}$ provided that:

(i) $f(x, y)$ is continuous function in $D \times D$;

(ii) $f(x, y)$ is Lipschitz continuous with respect to y:

$$|f(x, y_1) - f(x, y_2)| \leqq A\,|y_1 - y_2| \tag{6.9}$$

(iii) $f(x, y)$ is bounded:

$$\sup_{(x,y) \in D \times D} |f(x, y)| < \infty. \tag{6.10}$$

For the nonlinear differential equation (6.1) the function

$$f(x, y) = a x^{m(k-\alpha)-k} y^m + b x^{-k}, \tag{6.11}$$

satisfies the conditions (i) and (iii), provided that

$$0 < d < \infty, \ m(k - \alpha) - k \geqq 0, \ k \leqq 0, \tag{6.12}$$

and the condition (6.9) in (ii) be satisfied only for $m > 1$. Therefore from (6.12) we obtain that

$$\frac{m - \alpha}{m - 1} \leqq k \leqq 0, \tag{6.13}$$

which is impossible.

Thus the uniqueness problem of the solution (6.3) of the boundary value problem (6.1) - (6.2) is open. By the same situation such a problem is still also open for the solution (6.6) of the boundary value problem (6.4) - (6.5).

Acknowlegement

The work was initiated during the first author's visit to Fukuoka University as Research Fellow in 1995.

References

[1] S.N. Askhabov, Integral equations of convolution type with power nonlinearity, *Colloq. Math.* **62**(1991), 49-60.

[2] S.N. Askhabov and M.A. Betilgireev, Nonlinear convolution type equations, in *Seminar Analysis, Operator Equations and Numerical Analysis*, 1989/90 (Karl Weierstrass Institut für Mathematik, Berlin), 1990, 1-30.

[3] S.N. Askhabov and M.A. Betilgireev, Nonlinear integral equations of convolution type with almost decreasing kernels, *Diff. Equat.* **27**(1991), 234-242.

[4] S.N. Askhabov, N.K. Karapetyants and A.Ya. Yakubov, A nonlinear equation of convolution type (Russian), *Differentsial'nye Uravneniya* **22**(1986), 1606-1609.

[5] S.N. Askhabov, N.K. Karapetyants and A.Ya. Yakubov, Integral equations of convolution type with a power nonlinearity and their systems, *Soviet Math. Dokl.* **41**(1990), 323-327.

[6] P.J. Bushell, On a class of Volterra and Fredholm non-linear integral equations, *Math. Proc. Camb. Phil. Soc.* **79**(1976), 329-335.

[7] P.J. Bushell and W. Okrasinski, Nonlinear Volterra integral equations and the Apery identities, *Bull. London Math. Soc.* **24**(1992), 478-484.

[8] J. Gonserzewicz, H. Marcinkowska, W. Okrasinski and K. Tabitz, On the perlocation of water from a cylindrical reservoir into the surrounding soil, *Zastosow Math.* **16**(1978), 249-261.

[9] R. Gorenflo and S. Vessela, *Abel Integral Equations, Analysis and Applications*, (Lecture Notes in Math. 1461), Springer, Berlin, 1993.

[10] N.K. Karapetyants, On a class of nonlinear equations of convolution type (Russian), *Sci. Proc. Jubilee Sem. Bound. Value Problems*, Minsk, 1985, 158-161.

[11] N.K. Karapetyants, A.A. Kilbas and M. Saigo, On solution of nonlinear Volterra convolution equation with power nonlinearity, *J. Integral Equations and Appl.*, (to appear).

[12] N.K. Karapetyants, A.Ya. Yakubov, A convolution equation with power nonlinearity of negative order, *Soviet Math. Dokl.* **44**(1992), 517-520.

[13] J.J. Keller, Propagation of simple nonlinear waves in gas field tubes with friction, *Z. Angew. Math. Phys.* **32**(1981), 170-181.

[14] A.A. Kilbas, A nonlinear Volterra integral equation with a weak singularity, *Notes of Russian People's Friendship University* **2**(1994), 210-224.

[15] A.A. Kilbas and M. Saigo, On asymptotic solutions of nonlinear and linear Abel-Volterra integral equations, Investigations into Jack's lemma and related topics (Japanese) (Kyoto, 1994), *Sūrikaisekikenkyūsho Kōkyūroku* **No. 821** (1994), 91-111.

[16] A.A. Kilbas and M. Saigo, On solution of integral equation of Abel-Volterra type, *Differential Integral Equations* **8**(1995), 993-1011.

[17] A.A. Kilbas, M. Saigo and R. Gorenflo, On asymptotic solutions of nonlinear Abel-Volterra integral equations with quasipolynomial free term, *J. Frac. Calc.* **8**(1995), 75-93.

[18] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[19] W. Okrasinski, On the existence and uniqueness of nonnegative solution of a certain nonlinear convolution equation, *Ann. Polon. Math.* **36**(1979), 61-72.

[20] W. Okrasinski, On a nonlinear convolution equation occuring in the theory of water perlocation, *Ann. Polon. Math.* **37**(1980), 223-229.

[21] W. Okrasinski, Nonlinear Volterra equations and physical applications, *Extracta Math.* **4**(1989), 51-80.

[22] M. Saigo and A.A. Kilbas, On asymptotic solutions of nonlinear and linear Abel-Volterra integral equations II, Investigations into Jack's lemma and related topics (Japanese) (Kyoto, 1994), *Sūrikaisekikenkyūsho Kōkyūroku* **No. 821** (1994), 112-129.

[23] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon (Switzerland) *et alibi*, 1993.

[24] W.R. Schneider, The general solution of a non-linear integral equation of convolution type, *Z. Angew. Math. Phys.* **33**(1982), 140-142.